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ABSTRACT

In educational settings, it is becoming common for members of groups to be forced to choose among more than two candidates, programs, or objectives. When no majority favors any one of three or more alternative choices, there are several rational, although arbitrary, ways to proceed. The first attempt at making the choice ought to adopt the Condorcet Principle; that is, to vote on paired-comparisons with the loser in each of the sequential paired votings leaving the contest. If no majority winner emerges, then the Borda rule should be followed; with this technique, all voters assign weights to all of their choices in order of their preferences. Although this two-step procedure has some weaknesses, it is the best procedure available for finding the most acceptable choice. (Author/JM)

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THE PROBLEM OF DECIDING WITHOUT A MAJORITY: THE PARADOXES OF CHOICE

Walter J. Foley

Choosing, making a decision on one of several competing alternatives, is a task we face every day without much thought. After all, it isn't too difficult to reach a majority of one and decide. But it is becoming common for educators, as members of groups, to choose the "best" from among several competing candidates, programs, and/or objectives without the benefit of being a majority of one. Group settings for decisions are also assumed in many of the descriptions of concepts like "team administration," "collective negotiations," "board policy," and "needs assessment."

It is the position of this paper that when several persons are involved, the probability of a majority favoring any one choice decreases. It is also contended that "group participation" provides a context that increases the need to choose without a majority favoring any one of the possible alternatives. For example, a school board composed of seven members has to decide on one candidate from among three candidates in order to fill a position. A majority of the board members does not favor any one of the available candidates and the candidate with a plurality is chosen. That is to say that the board members chose the candidate that had received three first-place votes. Some board members felt this outcome was unsatisfactory as a majority of the members favored two other candidates (combined, they received four first-place votes). It seemed wrong to offer an important position to a candidate who was not the first choice of a majority of the board members.

The question posed in the often-heated discussion was: "What else can we do?" We have to make a decision.

In a search for the answer to this question, I encountered and became fascinated by the solutions that have been posed for the problem of choosing without a majority. The so-called paradoxes of choice were encountered when the solutions proposed had been tried. The paradoxes enter whenever an alternative—such as our choice of the plurality candidate—is employed in order to decide without a majority. Since it is now common to have group participation in a wide variety of decision settings, it is becoming increasingly important that a choice be made which the majority favors. When a majority cannot be found, it is even more important to know the limits of our choices. It is important to be aware of the possible paradoxes when choosing without a majority.

It will be shown that when it is impossible to obtain a majority in favor of any one alternative, the decisions reached by employing the strategies suggested as alternatives to a majority are all arbitrary. When a group doesn't favor one of the available choices, all they have done when they employ an alternative method of deciding is to choose that which looks rational but is actually arbitrary.

The First Discovery

During the French Enlightment, Jean-Charles de Borda[1] presented the first of several alternatives that have been posited. The number employed in his example will be given in an effort to be faithful to the original discussion of the paradox.

Assume that 21 voters are given the task of choosing from three available candidates for a position and a majority does not favor any one of the candidates. (Even though the number of voters is larger, this problem is not unlike the earlier school board problem of choosing a superintendent.) Eight of the voters are known to prefer candidate A to either candidate B or C. The thirteen remaining voters are known to prefer candidate B to A *and* to prefer candidate C to candidate A.

In Borda's example, the 13 voters who preferred either B *or* C to A divided their votes in such a way that seven of the voters were known to have placed B over C and the remaining six voters preferred C to B. The count of the first-place votes showed: eight votes for A, seven votes for B, and six votes for C. Since a majority in favor of one candidate did not exist (it would require 11 votes), candidate A was chosen the winner on the basis of simple plurality.

Borda, like some members in our school board illustration, felt this outcome was unsatisfactory. His reasoning was that since *clearly* B *or* C was preferred by more voters (N=13) than was A (N=8), any result that had A as the winner was unsatisfactory. The outcome, when simple plurality was employed in deciding, had gone against his concept of "the will of the majority." To correct this problem, Borda proposed a voting system that would provide an outcome that reflected "the will of the majority." The solution proposed has voters assign weights to all of their choices in the order of their preferences. It is, in essence, a ranking system.

Voters order the three candidates consistent with their choice preferences by giving three points to their first choice, two points to their second choice, and one point to the third choice. The points awarded by the voters to each candidate are then added. A summary of the original votes and the Borda point assignments is shown in table 1. Candidate B is the Borda winner with 45 points. Candidate C is now second with 44 points and candidate A, the original winner, is last with 37 points.

The second outcome of the voting process is consistent with Borda's "ideal" voting procedure, that is, one that reflects "the will of the majority." It should be noted that Borda's system was criticized on the grounds that the voters could misrepresent their preferences and the outcome of the pro-

Table 1

The Election Results with the Aplication of Borda Weights

*Voter Preferences**	*Candidates* A	B	C	*Point Totals*
Voters (N=8 for A)	24	12	12	48
Voters (N=7 for B)	7	21	14	42
Voters (N=6 for C)	6	12	18	36
VOTE TOTALS	37	45	44	126

*When each voter has 6 points to distribute as:
first place receives 3 points and is underlined;
second place receives 2 points;
third place receives 1 point.

cedure could be unduly influenced and thus distort "the will of the majority." In reply to this criticism, Borda asserted that his system was intended only for "honest men." In spite of the criticism, the Borda alternative to simple plurality has been widely adopted and has come to be known as the *Borda Rule*. In cases where three or more voters have the task of choosing between three or more alternatives, many people still invoke the rule. Today, this condition (multiple voters—multiple choices) is a common circumstance in educational settings and one would expect that the Borda Rule is often employed.

A close friend of Borda's, the Marques de-Condorcet,[2] was one of the critics of Borda's Rule. His concern was that a candidate can win an election without even receiving one first-place vote. Condorcet proposed a second method of choosing when there are several alternatives, and no clear majority winner. Condorcet's method puts the alternatives up for vote two at a time. Condorcet's position was that when there is no simple overall majority winner, the winner is the candidate (alternative) that wins over each of the other candidates (alternatives) in a pair-wise majority voting contest with the loser in each of the sequential paired votings leaving the contest. This alternative is also widely employed and is known as the *Condorcet Principle*.

To illustrate, reexamine the information shown in table 1. In any pairing of the three candidates that included A, candidate A would receive the most first-place votes and the other candidate would drop out. The third candidate would then be pitted against A and could also be expected to lose. Since a simple majority does not exist in the example (remember it would require eleven or more votes), the solution to the problem of plurality voting first posed by Borda differs from the solution that might be found under the Condorcet Principle. The question of the "rightful winner" that was thought to be answered by Borda remained. The new question became: Which voting alternative was best?

The answer is not clear. Both the Borda and the Condorcet solutions contain the possibility of the occurrence of separate paradoxes. The Borda Rule allows for the occurrence of a winner that was no one's first choice. The Condorcet paradox ("I'effect Condorcet") states that the eventual winner is determined by the process. The paradox can occur when three or more judges make three or more orderings of choice and do so in such a way that the ultimate winner is determined by where one begins the pairing of the choices. As an example, suppose that three teachers are on a curriculum committee and they are asked to order their preferences from among three competing textbooks (A, B, and C). They do so as follows:

Teacher 1	ABC
Teacher 2	BCA
Teacher 3	CAB

The reading of the preference orders is that for teacher 1, textbook A is preferred to B *and* textbook B is preferred to C; for teacher 2, textbook B is preferred to C *and* textbook C is preferred to A, and so forth. Obviously, the Borda Rule of assigning points (such as 3, 2, 1) to the order of the choices made by the teahers would result in a tie since the orderings all would have the same total. Applying the Condorcet Principle of pairwise comparisons to the teacher data would show that for each choice pair:

A beats B by 2 to 1 (Teachers 1 and 3 prefer A; teacher 2 prefers B)
B beats C by 2 to 1 (Teachers 1 and 2 prefer B; teacher 3 prefers C)
C beats A by 2 to 1 (Teachers 2 and 3 prefer C; teacher 1 prefers A).

The paradox works like this. Say that textbook A is first pitted against B; the outcome is that A wins (from the above: A beats B by 2 to 1) and textbook B is eliminated. Then, according to the procedure, A is next pitted against textbook C. The outcome is that C wins and A is eliminated. Textbook C is chosen and the voting terminates. Consider what would have happened had we decided to start the procedure with textbook B. When textbook B is first pitted against C, the outcome is that B wins (from the above: B beats C by 2 to 1) and textbook C is eliminated. Next B is paired with textbook A and A eliminates B. The winner is textbook A. It is clear that either textbook A or B or C can win since the results are entirely dependent on the choice of the first pairing. *Where* we start the process of paired comparison determines the final outcome. Thus, the Condorcet paradox.

After the solutions to the problem of deciding without a majority were first pointed out and alternative solutions were offered by Borda and Condorcet, little attention to the occurrence of voting paradoxes was found in the literature for almost 90 years.

The Second Discovery

Another attempt at a solution to this problem occurred in England when C. H. Dodgson[3] (Lewis Carroll) addressed the question of choosing without a majority in the context of selecting a new dean for Christ College. He felt he had solved what he later discussed as the problem of "cyclical majorities." When Dodgson made his proposal, he first explained the shortcomings of each of what he termed "unsatisfactory" solutions. The methods he considered and rejected included: Voting on all alternatives at the same time and eliminating the least-supported alternative; assigning weights to choices (á la Borda); entering into a series of "two-at-a-time" preference contests (á la Condorcet); having the possible alternatives nominated, seconded, and then voted upon.

Dodgson presented his solution to (but apparently unknown to him) the Condorcet paradox in a paper dated June 15, 1874. He defined the problem as that of "cyclical majorities" (p. 277). To solve the problem, one computes the choice that required the fewest number of inversions (changes in preference order by those voting) to obtain a winner. Then, *after the consent of the voters who would make the inversions was granted,* the winner would be declared.

Just how the Dodgson solution would work for our 21-voter Borda example follows: For candidate A to win, a minimum of three B voters would have to change or invert their first choice. Candidate A would be declared the winner since this choice would require the fewest inversions to obtain a majority, after permission had been granted. It should be noted that this need not be the case (that A wins), and that the solution becomes more complicated for double inversions (from first place to third place) and might not work at all for the choice orders in the three-teacher, three-textbook Condorcet paradox example.

The uniqueness of Dodgson's mind was reflected by a consideration of "no election" as a voting choice. He felt that groups should have the choice of not deciding. The choice of "no election," if it failed to receive a majority of the votes, was also seen as an unsatisfactory outcome since it becomes one of the several possible less-than-favored choices. Dodgson also included the choice of terminating the voting process with the question: "Shall the successful candidate be elected, or shall there be no election?" This "Dodgson" choice of no election is

still made quite often as evidenced by position notices that read: "The search has been reopened for the position of _____." Obviously, a choice of no choice had been made from the original list.

Dodgson gave the following advice when he was asked if voters were bound in honor to vote for their *real* preference in a paired-comparison (Condorcet type) situation (to quote): "I think it desirable that all should know the rule by which this game may be won. It is simply this: 'In any division taken on a pair of issues neither of which you desire, vote against the most popular'." His advice was more realistic than Borda's earlier response to a similar question.

In summary, three main solutions were presented as alternative ways to proceed when faced with a situation in which a group is faced with a choice from more than two alternatives and no clear majority exists for an alternative. The Borda solution assigns points to each of the preferences with the highest point total winning. The Condorcet solution is that the choice supported in each two-at-a-time comparison of the alternatives continues until the ultimate winner emerges. The Dodgson solution is that the choice requiring the fewest number of changes in the original choice order is declared the winner after consent is granted by those who must make the inversions.

Little attention was paid to the problem of choosing a winner without a majority until it surfaced again as a challenge to the method used in making the social choices in the "New Frontier." It was the 1950s that provided the environment for what we may call "a modern discussion" of the paradoxes of choice.

The Modern Discussion

Most current discussions of the paradoxes are heavily based on statistical, probabilistic, and set theory notions. The reader who is interested in these technical presentations and their mathematical assumptions is provided with several important references in the bibliography. Here we are concerned with how the paradoxes add to our understanding of the inherent limitations when forced into educational choices without a clear group majority.

The modern (post 1950) treatment of the paradoxes begins with the definitions. When we say that preferences exist between alternatives, it is assumed that: first, there exists a set of decision alternatives that is non-empty and finite and that the choices open to the decision makers lie within that set; second, the preference of each individual (i) is such that if i prefers A over B (ApB) and also prefers B over C (BpC), then i also prefers A over C. For i, if ApB *and* BpC, then ApC. This relationship among choices is called *transitivity*. The modern discussions of preference orders that concern us assume this property.

When points or weights are assigned to choices, as in ranking, they are assumed to have the same directional value. That is, the points assigned are said to be *monotonic*. For example, the weights (3,2,1) in the Borda system are monotonic. It was interesting that Borda originally considered nonmonotonic scoring weights as he attempted to find a solution to the voting problem and rejected the idea because he felt that they capitalize on chance. The early developers of attitude instruments like the Minnesota Teacher Attitude Inventory employed nonmonotonic scoring weights and have, after bitter experience, come to share the Borda conclusion.

More current discussions of the problem of choosing from among several competing social policies (that is, finding a solution for the problem of constructing social priorities), define the task as that of determining a clear "Social Choice Function" (SCF). In the revived interest, the early voting solutions of Borda, Condorcet, and Dodgson have been, in a sense, rediscovered. As a result of our more recent interest in the SCF problem, Kenneth Arrow, while working in the field of welfare economics, concluded that—without a majority—it is impossible to construct a "clear" SCF to serve as a focus for welfare program efforts.[4] In an educational setting, a parallel would be the impossibility of choosing a curricular focus when no clear majority preference emerged for any of the choices presented after conducting a "needs assessment."

Arrow's discussion of the problem of choosing the "best" of the competing alternatives without a majority choice is considered a classic in the field. He presented what he called the *Possibility Theorem*. By beginning with two axioms (that the three or more possible choices are connected and that they are transitive), Arrow concluded that: ". . . if, in voting on persons or motives, transitivity is not required and if in fact no majority exists for one alternative, then the social preference is entirely dependent on the order in which candidates or motions are voted on" (p. 905). This is, in essence, a restatement of the Condorcet paradox when applied to situations with large numbers of voters and/or choices.

Several years later, Black,[5] in effect, extended Arrow's pessimistic position on the impossibility of finding a solution to the problem in his definitive work on the functioning of committees and the election process. Black was concerned with the likelihood of success of formal motions made in the context of the order of voting on the motions. "The later any motion enters the voting, the greater its chance of adoption" (p. 40). Riker[6] also addressed the paradoxes of choice in 1961 and concluded that the solutions proposed contain more difficulties than do the paradoxes. For a current discussion of the Riker position consult the Niemi and Riker (1976) reference.

In a study of the possibilities of still other paradoxes conducted with large numbers of voters/issues through use of computer simulations, Fishburn[7] found (under the assumptions of set theory) four more paradoxes. The four other paradoxes (given the proposition that large numbers of voters are presented with many alternatives) are:

> *The "dominated winner" paradox:* In sequential paired-comparison voting (Condorcet, Dodgson) a choice (X) "Dominates" * another choice (Y), yet the dominated choice (Y) wins. In Fishburn's simulations, this "impossible" result occurred once in 12,000 times.
>
> *The "inverted order" paradox:* When the winner X (chosen by a linear order of Borda point totals) is removed and the process is repeated for the remaining alternatives, the alternatives follow the exact opposite order from the original order. The likelihood of the occurrence of this paradox increases as the number of voters increases.
>
> *The "winner-turns-loser" paradox:* The winner in one situation loses in every other situation but one that involves it and exactly one other alternative, and is the loser in every other situation that involves it and at least one (but not all) of the other alternatives. The likelihood of the original Borda Rule winner becoming a loser when one of the losers is removed from the voting increases as the number of voters increases (the N of alternatives being a fixed number).
>
> *The "truncated point-total" paradox:* When less than the total possible ranks are employed in the voting, it is possible for each separate truncation (reduction in possible ranks) to produce a unique winner. When the number of alternatives is ten, any of the nine of the ten alternatives can win depending on the truncating procedures adopted.

Discussion

Those who find themselves members of "consensus" groups will never have to become concerned with these paradoxes. Consensus groups (by definition) never vote and they never have to admit to situations that would allow for voting

*"Dominates" refers to what is termed the Pareto condition. This condition says that the SCF function both socially and individually preferred has the same meaning for any profile.

paradoxes. The strength of consensus is in the belief that it is possible. Voting is seen as an impediment to that process. The "political realists" seem to know intuitively that, without a majority, a "right" choice does not even exist, let alone attain discernibility. They prefer to operate within the contexts of the paradoxes and to manipulate the voting process in ways that often serve to keep the source of their power hidden. They know the rules and use them to their advantage. Last, for those who consider themselves "rational," the paradoxes are often confronted. Since they prefer to believe that behavior is rational, that a "right" alternative exists, and that it is possible to choose this alternative through some democratic process such as paired comparisons, ranking, and so forth, they vote. This is to say that members of "rational groups" tend, after a fashion, to seek a procedure that will produce a winner even when a majority cannot be found for one of the alternatives. But they are simply wrong in this belief. I wish them my best since they are forced to live within the paradoxes of choice.

Conclusion

The question remains: What can be done when a person or group is forced to make a choice from among three or more alternatives when three or more voters cannot reach a majority on one of the alternatives? The answer, after all of the discussion, is that there is no clear answer to this question. But it is the author's contention that a procedure that is neutral, consistent, and faithful to the intent of the decision situation should be chosen. After all, we do have to decide. The advice offered is that when faced with situations where a choice is necessary and there is no clear majority preference for one of the alternatives: First, adopt the Condorcet Principle; if no majority winner emerges as a result of implementing this process, then follow the Borda Rule. This advice is similar to that given by Black in 1958 (pp. 66-67) but in a different context. It still appears to be the best advice available as a procedure to make choices in educational settings under the conditions presented.

It should be pointed out that the ordering of preferences within the alternatives under the Borda Rule (the second step) is *weak* since under the application of the simple paired-comparisons test of the Condorcet Principle (the first step) no winner emerged. Still, it should be noted that the intent of a preference decision (when made under the constraint of multiple-choice possibilities and voting with no clear majority winner) is to find a choice that is most acceptable. The two-step procedure proposed is, if not unassailable, the most defensible of those available. In general, when a majority does not favor one of the available choices, you are in a situation that does not have a clear solution.

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